

3. The Point Kinetics Equations

- When a perturbation is made to the reactor properties, the steady state no longer holds, and the evolution of the neutron flux must be obtained from the time-dependent diffusion equation.
- The effect of local perturbations on $\phi(r,E,t)$ will quickly propagate. In many cases, there is a slight re-adjustment of the flux shape in a few milliseconds, after which the global flux level will increase or decrease, depending on whether the perturbation has increased or decreased k_{eff} .
- When the flux shape varies little or slowly, one can predict accurately the evolution of the *total reactor power* as a function of the changes brought by the perturbation to the *average* reactor properties, neglecting completely the shape changes. This is *the point-kinetics approximation*.

$$\frac{1}{v} \frac{\partial \phi}{\partial t}(r,E,t) = (F_p - M) \phi(r,E,t) + S_d(r,E,t) + S(r,E,t)$$

with:

$$F_p \phi = \chi_p(E) \int_0^\infty v_p \Sigma_f(r,E',t) \phi(r,E',t) dE'$$

$$M \phi = -\vec{\nabla} \cdot D \vec{\nabla} \phi + \Sigma_a \phi - \int_0^\infty \Sigma_s(r,E \rightarrow E',t) \phi(r,E',t) dE'$$

$$S_d = \sum_k \chi_{dk}(E) \lambda_k C_k(r,t)$$

$$S = \text{arbitrary independent source}$$

with:

$$\frac{\partial C_k}{\partial t} = -\lambda_k C_k(r,t) + \int_0^\infty dE' v_{dk} \Sigma_f(r,E',t) \phi(r,E',t)$$

$$(k = 1, 2 \dots K)$$

Flux Factorization

- In order to take advantage of the fact that the flux shape can vary more slowly than the amplitude, we introduce the following factorization:

$$\phi(r, E, t) = p(t) \psi(r, E, t)$$

- Substitution of this factorized form in the diffusion equation leads to a new equation for the shape function ψ . After dividing by $p(t)$, we find

$$\frac{1}{v} \left(\frac{1}{p(t)} \frac{dp}{dt} \psi(r, E, t) + \frac{\partial \psi}{\partial t}(r, E, t) \right) = [F_p - M] \psi(r, E, t) + \frac{1}{p(t)} S_d(r, E, t)$$

- This equation will be coupled to a second equation for the amplitude $p(t)$.
- This formulation is quite general. The advantage is that the shape ψ varies much more slowly with t than the amplitude $p(t)$. This permits *an integration step for ψ much larger than that used to calculate the amplitude p* . The separate solution of the shape and amplitude equations can lead to substantial machine-time savings.
- The equation for the amplitude is obtained by integrating the diffusion equation over the domain (r, E) . Since every point in the domain may not have the same importance, we first multiply the diffusion equation by an *arbitrary time-independent weighting function $w(r, E)$* , defined over the same domain as ϕ .

Amplitude Equation

- In analogy with the steady-state production operator, we define the following operator F in the time-dependent case:

$$F\phi(r, E, t) = (F_p + F_d)\phi(r, E, t)$$

- The operator F_p represents the *instantaneous source of prompt neutrons* (at time t). With the previous factorization, we get

$$\begin{aligned} F_p\phi(r, E, t) &= p(t) \chi_p(E) \int_0^\infty dE' \nu_p \Sigma_f(r, E', t) \psi(r, E', t) \\ &= p \cdot F_p \psi \end{aligned}$$

- The operator F_d *does not* represent the instantaneous source of delayed neutrons. It is the source of delayed neutrons which would be found at equilibrium at (r, E) if the reactor were in steady-state with the instantaneous flux $\phi(r, E, t)$.

$$\begin{aligned} F_d\phi(r, E, t) &= p(t) \sum_k \chi_{dk}(E) \overbrace{\int_0^\infty dE' \nu_{dk} \Sigma_f(r, E', t) \psi(r, E', t)}^{\text{Spatial distribution of the precursor creation rate}} \\ &= p \cdot \sum_k F_{dk} \psi \\ &= p \cdot F_d \psi \end{aligned}$$

- Substituting in the diffusion equation, we find:

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = (F - M - F_d)\phi + S_d + S$$

- Introducing the factorization $\phi = p\psi$, multiplying by an arbitrary weight function $w(r, E)$ and integrating over the domain, we find the following equation for the amplitude $p(t)$:

$$\left\langle w, v^{-1} \psi \right\rangle \cdot \frac{dp}{dt} + \left\langle w, v^{-1} \frac{\partial \psi}{\partial t} \right\rangle \cdot p = \left\langle w, (F - M - F_d) \psi \right\rangle \cdot p + \left\langle w, S_d \right\rangle + \left\langle w, S \right\rangle$$

Amplitude Equation (cont'd)

- Delayed-neutron source term contains a weighted sum of precursors:

$$\langle w, S_d \rangle = \sum_{k=1}^K \lambda_k \langle w, \chi_{dk} C_k \rangle$$

- pre-multiplying the precursor equation by $\chi_{dk}(E)$, multiplying the result by $w(r, E)$, and integrating over the domain, we find:

$$\frac{d\langle w, \chi_{dk} C_k \rangle}{dt} = \langle w, F_{dk} \psi \rangle \cdot p(t) - \lambda_k \langle w, \chi_{dk} C_k \rangle$$

- We now define the following scalar quantities to simplify the notation:

$$c_k(t) = \frac{\langle w, \chi_{dk} C_k \rangle}{\langle w, v^{-1} \psi \rangle}$$

$$\xi_\psi(t) = \frac{\langle w, v^{-1} \frac{\partial \psi}{\partial t} \rangle}{\langle w, v^{-1} \psi \rangle}$$

$$s_e(t) = \frac{\langle w, S \rangle}{\langle w, v^{-1} \psi \rangle}$$

- We find:

$$\frac{dp}{dt} = (\alpha_p(t) - \alpha_\beta(t) - \xi_\psi(t)) p(t) + \sum_{k=1}^K \lambda_k c_k(t) + s_e(t)$$

and

$$\frac{dc_k}{dt} = \alpha_{\beta_k}(t) \cdot p(t) - (\lambda_k + \xi_\psi(t)) \cdot c_k(t)$$

$$(k = 1, 2, \dots, K)$$

Amplitude Equation (cont'd)

- The following *kinetics parameters* have been introduced:

$$\alpha_p(t) = \frac{\langle w, (F - M) \psi \rangle}{\langle w, v^{-1} \psi \rangle}$$

and

$$\begin{aligned} \alpha_\beta(t) &= \sum_{k=1}^K \alpha_{\beta_k}(t) \\ &= \sum_{k=1}^K \frac{\langle w, F_{dk} \psi \rangle}{\langle w, v^{-1} \psi \rangle} \end{aligned}$$

- We note that *no approximation has been made to this point*.
- In place of the time-dependent diffusion equation, coupled to the precursor equations, we have after factorization a system of coupled equations consisting of an equation for the shape function and the amplitude equations. These two systems are entirely equivalent.
- In the *quasi-static method*, the shape equation is solved over macro-intervals, within which the amplitude equations are separately integrated, using a known shape. The factorization thus introduces an additional degree of freedom, which can be used to simplify the numerical solution of the time-dependant diffusion equation.

The Normalization Constraint

- We note also that *the relative normalization of p and ψ is arbitrary*, since both these quantities are functions of time. In the absence of a normalization constraint, all solutions of the following form must be allowed:

$$\begin{aligned}\phi(r, E, t) &= p(t) \cdot \psi(r, E, t) \\ &= \left(p(t) \cdot a(t) \right) \cdot \left(\frac{\psi(r, E, t)}{a(t)} \right) \\ &= \tilde{p}(t) \cdot \tilde{\psi}(r, E, t)\end{aligned}$$

- In order to make the relative normalization unique and to transfer as much as possible of the time variation from ϕ to the amplitude, we will impose:

$$\langle w, v^{-1} \psi \rangle = \left\langle \frac{w(r, E) \psi(r, E, t)}{v(E)} \right\rangle = K_0$$

where K_0 is an arbitrary constant. As a consequence, we note:

$$\xi_{\psi}(t) = 0$$

- The amplitude equations then become:

$$\frac{dp}{dt} = [\alpha_p(t) - \alpha_\beta(t)] p(t) + \sum_{k=1}^K \lambda_k c_k(t) + s_e(t)$$

and

$$\frac{dc_k}{dt} = \alpha_{\beta_k}(t) \cdot p(t) - \lambda_k c_k(t)$$

($k = 1, 2, \dots, K$)

The Point Kinetics Approximation

- The normalization constraint does not constitute an approximation:

The simultaneous solution of amplitude equations together with the above normalization constraint on ψ , affects the value of ρ but not of the product $\phi = \rho\psi$, which is conserved.

- The solution of ϕ is independent of the choice for the weighting function w :

The use of an arbitrary weight function gives an additional degree of freedom. We may then select a particular function W to best serve our purpose.

OBSERVATIONS:

- The amplitude equation for $\rho(t)$ is a set of ordinary differential equations, much easier to solve than the shape equation (for $\psi(t)$) or the original diffusion equation (for $\phi(t)$), which are second order partial differential equations.
- The amplitude equation yielding $\rho(t)$, a quantity proportional to the total neutron power, is controlled by only two kinetics parameters, namely $\alpha_\rho(t)$ and $\alpha_\beta(t)$, which depend on the *nuclear properties* of materials within the domain.

OBJECTIVE:

Our objective is to decouple the amplitude equation from the shape equation, so that we can estimate simply the *reactor response to localized perturbations*.

The Point Kinetics Approximation (cont'd)

APPROXIMATION:

The central approximation in point kinetics is to assume that the shape function ψ used to calculate the kinetics parameters is constant. It is usually equal the initial steady-state flux distribution.

- The initial steady-state flux distribution, $\phi_o(r,E)$, is normally obtained by solving (once) *the static diffusion equation* with a detailed model of the reactor.
- If, in reality, the localized perturbation does not affect significantly the flux shape, we can expect that the solution to the point kinetics approximation, $p(t)$, will provide a reasonably accurate prediction of the total reactor power transient.
- With the introduction of this approximation, we now allow a difference between the approximate flux shape, ϕ_o , used to calculate the amplitude, and the real time dependant flux shape, ψ . It will thus be advantageous to choose a weighting function which minimizes the error in $p(t)$ due to the uncertainty in the shape, $\delta\psi = \phi_o - \psi$.

Conventional Form of the Point-Kinetics Equations

(Initially Critical Reactor)

- At initial steady state, we have:

$$\sum_{k=1}^K \lambda_k c_k(0) = \frac{\beta}{\Lambda}(0) \cdot \rho(0)$$

$$\mathbf{M}_0 \phi_0 = \lambda_0 \mathbf{F}_0 \phi_0$$

where, in principle, $\lambda_0 = 1$ since the reactor is assumed critical.

- We shall choose the initial adjoint flux, ϕ_0^* , for the weighting function:

$$w(r, E) = \phi_0^*(r, E)$$

Reason: the error committed as a result of using the initial shape to calculate the kinetics parameter is of second order in the shape error, as long as the initial adjoint flux is used as weighting function.

- While the amplitude equation contains in reality only two kinetics parameters, the conventional form of the point-kinetics equations shows three parameters, by explicitly including the *dynamic reactivity* as one parameter in the equation.
- Dynamic reactivity is defined, by analogy to the static reactivity introduced earlier:

$$\rho(t) = \frac{\langle \phi_0^*, (\mathbf{F} - \mathbf{M}) \phi_0 \rangle}{\langle \phi_0^*, \mathbf{F} \phi_0 \rangle}$$

The Point Kinetics Parameters

- Multiplying and dividing the first kinetics parameter yields:

$$\alpha_\rho(t) = \underbrace{\frac{\langle \phi_0^*, (\mathbf{F} - \mathbf{M}) \phi_0 \rangle}{\langle \phi_0^*, \mathbf{F} \phi_0 \rangle}}_{\rho(t)} \cdot \underbrace{\frac{\langle \phi_0^*, \mathbf{F} \phi_0 \rangle}{\langle \phi_0^*, \mathbf{V}^{-1} \phi_0 \rangle}}_{1/\Lambda(t)}$$

- We have in this way separated the parameter into two components, the dynamic reactivity $\rho(t)$ and a new parameter, $\Lambda(t)$, which we shall call the mean prompt-neutron lifetime:

$$\Lambda(t) = \frac{\langle \phi_0^*, \mathbf{V}^{-1} \phi \rangle}{\langle \phi_0^*, \mathbf{F} \phi \rangle}$$

- Treating second the parameter in a similar fashion, we find

$$\alpha_{\beta_k}(t) = \underbrace{\frac{\langle \phi_0^*, \mathbf{F}_{dk} \phi_0 \rangle}{\langle \phi_0^*, \mathbf{F} \phi_0 \rangle}}_{\beta_k(t)} \cdot \underbrace{\frac{\langle \phi_0^*, \mathbf{F} \phi_0 \rangle}{\langle \phi_0^*, \mathbf{V}^{-1} \phi_0 \rangle}}_{1/\Lambda(t)}$$

- We have in this way introduced the new parameter $\beta(t)$, which we shall call the effective delayed-neutron fraction:

$$\begin{aligned} \beta(t) &= \sum_{k=1}^K \beta_k(t) \\ &= \sum_{k=1}^K \frac{\langle \phi_0^*, \mathbf{F}_{dk} \phi \rangle}{\langle \phi_0^*, \mathbf{F} \phi \rangle} \end{aligned}$$

The Point Kinetics Parameters (cont'd)

- Dynamic reactivity $\rho(t)$ has no units. It is generally a small number, so that it is conventional to use the (non-physical) units mk , which are simply fractions of 0.001
- $\rho(t)$ varies rapidly and significantly when the reactor is perturbed (control rod movements, refuellings, temperature and density changes,...)
- On the other hand, over the short time horizon of the transients considered in reactor kinetics, we can often neglect the time variation of $\beta(t)$ and $\Lambda(t)$.
- Λ has units of time (s). It measures the length of time it takes for an average neutron to reproduce itself via the fission chain reaction (i.e. the neutron generation)
- β is the *effective* delayed neutron fraction. It is a weighted average and accounts for the non-uniform distribution of fissile material in the reactor and the different delayed neutron emission spectra.
- Typical values:

	CANDU (heavy water)	LWR (light water)	MAPLE (research)
Λ	$\sim 10^{-3}$ s	$\sim 10^{-4}$ s	$\sim 0.7 \times 10^{-4}$ s
β	~ 0.0059	~ 0.0075	~ 0.0082

The Point Kinetics Equations

- In the absence of an external source, the conventional form of the kinetics equations is therefore the following:

$$\begin{aligned} \frac{dp}{dt}(t) &= \left[\frac{\rho(t) - \beta}{\Lambda} \right] p(t) + \sum_{k=1}^K \lambda_k c_k(t) \\ \frac{dc_k}{dt}(t) &= -\lambda_k c_k(t) + \frac{\beta_k}{\Lambda} p(t) \quad (k = 1, 2, \dots, K) \end{aligned}$$

- An alternate form of the P.K. equations can be written, if we choose the following variable for the delayed precursor concentration, instead of $c_k(t)$

$$\zeta_k(t) = \frac{\langle \phi_0^*, \chi_{dk} C_k \rangle}{\langle \phi_0^*, \lambda_0 F_0 \psi_0 \rangle}$$

Then we have:

$$\begin{aligned} \Lambda \frac{dp}{dt}(t) &= [\rho(t) - \beta] p(t) + \sum_{k=1}^K \lambda_k \zeta_k(t) \\ \frac{d\zeta_k}{dt}(t) &= -\lambda_k \zeta_k(t) + \beta_k p(t) \quad (k = 1, 2, \dots, K) \end{aligned}$$

Point Kinetics (conclusions)

- The Point Kinetics equations are simple;
- They can be solved analytically in the case of constant reactivity;
- The PK parameters correctly characterize the reactor and can be obtained in a straight-forward manner:
 - rough estimate (survey calculations),
 - detailed estimate starting from a complete description of a reactor and a steady-state solution of the diffusion equation (engineering calculations);
- If reactivity varies (as it generally does during a transient), the PK equations are *easily solved numerically* (on personal computers);
- Point Kinetics calculations are very useful in **control and safety studies**, because *feedback effects can easily be modelled*;
- Because the *approach is approximate* (assumption of constant flux shape), caution must be used. Validation with detailed space-time kinetics methods (time-dependent diffusion equation in 3D).